MTH 406: Differential geometry of curves and surfaces

Homework I

(Due 05/02)

Problems for turning in

1. State and give formal mathematical proofs of the analogs of 1.2 (ix) and 1.2 (x) from the Lesson Plan for space curves.

2. If all the tangent lines of a regular parametrized curve γ pass through a single point, then show that γ is a part of a straight line. Does this conclusion hold if γ is not regular?

3. Show that if the curvature $\kappa(t)$ of a regular curve is positive everywhere, then $\kappa(t)$ is a smooth function of t. Show that this does not hold if the positivity of κ is not assumed.

4. Let γ be a unit-speed space curve. Show that if the normals of γ pass through a fixed point, then γ is a part of a circle.

5. Let γ be a unit-speed space curve with $\tau(s), \dot{\kappa}(s) \neq 0$, for all s. Show that α lies on a sphere if, and only if,
$$R^2 + (\dot{R})^2 S^2 = const,$$
where $R = 1/\kappa$ and $S = 1/\tau$.

Problems for practice

1. Let γ be a unit-speed plane curve with nowhere-vanishing curvature. We define the *center of curvature* $\epsilon(s)$ of γ at the point $\gamma(s)$ is defined by
$$\epsilon(s) = \gamma(s) + \frac{\eta(s)}{\kappa_{\pm}(s)}.$$
Prove that the circle with center $\epsilon(s)$ and radius $|1/\kappa_{\pm}(s)|$ is tangent to γ and $\gamma(s)$, and also has the same curvature as γ. (Note that this is called the *osculating circle.*)

2. Show that the signed curvature $\kappa_{\pm}(t)$ of a regular plane curve $\gamma(t)$ is a smooth function of t.

3. Show that the volume of a parallelopiped generated by three linearly independent vectors $u, v, w \in \mathbb{R}^3$ is given by $\|(u \times v) \cdot w\|$.

4. Plot the curve (helix) given by
$$\gamma(s) = (a\cos(s/c), s\sin(s/c), b(s/c)),\ s \in \mathbb{R},$$
where $c^2 = a^2 + b^2$, in Mathematica. Also, compute its curvature and torsion.

5. Plot the curve (*catenary*) given by
$$\gamma(t) = (t, \cosh(t)),\ t \in \mathbb{R},$$
in Mathematica. Show that its signed curvature is $1/\cosh^2(t)$.